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ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS: A SURVEY

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ABSTRACT

Consider solutions $(H(x,\varepsilon),G(x,\varepsilon))$ of the von Kármán equations for the swirling flow between two rotating coaxial disks

1.1) $$\varepsilon H^{iv} + HH''' + GG' = 0 ,$$

1.2) $$\varepsilon G'' + HG' - H'G = 0 .$$

In this survey we describe much of the activity of the past 30 years - involving physical conjecture, numerical computation, asymptotic expansions and rigorous mathematical results. In particular we focus on the questions of existence and nonuniqueness, monotonicity, and "scaling".

AMS (MOS) Subject Classifications: 34B15, 34E15, 35Q10

Key Words: Ordinary Differential Equations, Rotating Fluids, Similarity Solutions, Asymptotic Behavior, Existence, Nonuniqueness, Scaling, Computation, Matched Asymptotic Expansions

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SIGNIFICANCE AND EXPLANATION

Under appropriate conditions the steady-state flow of fluid between two planes rotating about a common axis perpendicular to them may be described by two functions $H(x,\varepsilon)$, $G(x,\varepsilon)$ which satisfy the coupled system of ordinary differential equations

$$\varepsilon H^{iv} + HH''' + GG' = 0$$

$$\varepsilon G'' + HG' - H'G = 0 .$$

The quantity $\varepsilon > 0$ is related to the kinematic viscosity and $\frac{1}{\varepsilon} = R$ is usually called the Reynolds number.

These equations have received quite a bit of attention. First of all, people who are truly interested in the phenomena modeled by these equations, e.g. fluid dynamicists, are interested in this problem. However, as these equations have been studied by a variety of mathematical methods, they have taken on an independent interest. The major methods employed have been (i) Matched Asymptotic Expansions and (ii) Numerical Computations. In both approaches technical problems have appeared. There may be "turning points," i.e. points at which $H(x,\varepsilon) = 0$. Such points require special and delicate analysis within the theory of (i). As numerical problems, these equations are "stiff" - precisely because ε is small. The occurrence of "turning points" only makes computation more difficult.

For these reasons, these equations have become "test" problems for methods of "matching in the presence of turning points" and "stiff O.D.E. solvers." However, when one has "test problems," one needs to know the answers. Unfortunately here the answers are largely unknown.

In the 60 years since the basic von Kármán paper (1921) and the 30 years since the famous papers of Batchelor (1951) and Stewartson (1953) there has been an intensive interaction between physically based conjecture, numerical calculations, formal asymptotic expansions and rigorous mathematical results. In this survey we discuss several specific questions and describe this interaction.

This article is an expansion of a talk given in Oberwolfach, Germany at the Conference on Singular Perturbations.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE SWIRLING FLOW BETWEEN ROTATING COAXIAL DISKS: A SURVEY

Seymour V. Parter

1. Introduction

In 1921 T. von Kármán [12] developed the similarity equations for axi-symmetric, incompressible, steady flow - "swirling flow". Let (q_r, q_θ, q_x) be the coordinates of velocity in cylindrical coordinates, (r, θ, x). von Kármán assumed that there is a function $H(x,\varepsilon)$ such that

$$q_x = -H(x,\varepsilon) .$$

Then (see [2], [12]) there is a function $G(x,\varepsilon)$ so that the velocity components are described by

$$q_r = \frac{r}{2} H'(x,\varepsilon), \qquad q_\theta = \frac{r}{2} G(x,\varepsilon) .$$

The functions $\langle H(x,\varepsilon), G(x,\varepsilon)\rangle$ satisfy the equations

$$\varepsilon H^{iv} + HH''' + GG' = 0 , \tag{1.1}$$

$$\varepsilon G'' + HG' - H'G = 0 . \tag{1.2}$$

The quantity $\varepsilon > 0$ *is related to the bulk viscosity. Equation* (1.1) *can be integrated to* yield

$$\varepsilon H''' + HH'' + \frac{1}{2} G^2 - \frac{1}{2} (H')^2 = \mu \tag{1.3}$$

where μ is a constant of integration.

In the case originally studied by von Kármán, the flow above a single disk, we have a problem on the infinite interval $[0,\infty]$ and the constant of integration is known, i.e.,

$$\mu = \frac{1}{2} \Omega_\infty^2$$

where $\Omega_\infty = G(\infty,\varepsilon)$. Moreover, in this case the parameter ε may be "scaled out". Assume $\Omega_0 \neq 0$ and let

$$\xi = x/\sqrt{\varepsilon} , \tag{1.4a}$$

$$H(x,\varepsilon) = \sqrt{\varepsilon}\, h(\xi), \qquad G(x,\varepsilon) = g(\xi) G(0,\varepsilon) . \tag{1.4b}$$

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Then, the functions $\langle h(\xi), g(\xi) \rangle$ satisfy

(1.5) $$h''' + hh'' + \frac{1}{2} g^2 - \frac{1}{2} (h')^2 = \frac{1}{2} (\Omega_\infty/\Omega_0)^2 ,$$

(1.6) $$g'' \quad hg' - h'g = 0 ,$$

and the boundary conditions

(1.7a) $h(0) = 0$, (no penetration) ,

(1.7b) $h'(0) = 0$, (no slip) ,

(1.7c) $g(0) = 1$, normalization ,

(1.7d) $h(\xi)$ bounded as $\xi \to \infty$,

(1.7e) $g(\xi) \to \Omega_\infty/\Omega_0 = \mu_\infty$, as $\xi \to \infty$.

If we consider the flow between two planes, $x = 0$, $x = 1$ rotating with constant angular velocities $\Omega_0/2$, $\Omega_1/2$, then the quantity $\mu = \mu(\varepsilon)$ is unknown. This latter case was first studied by Batchelor [2] and Stewartson [38] who gave conflicting arguments and conjectures. In this case the boundary conditions are

(1.8a) $H(0,\varepsilon) = H(1,\varepsilon) = 0$, [no penetration]

(1.8b) $H'(0,\varepsilon) = H'(1,\varepsilon) = 0$, [no slip]

(1.8c) $G(0,\varepsilon) = \Omega_0$, $G(1,\varepsilon) = \Omega_1$, $|\Omega_0| + |\Omega_1| \neq 0$.

Both of these problems have been the subject of many numerical studies and have been attacked by formal matched asymptotic expansions. The von Kármán problem was studied numerically by D. M. Hannah [8] in 1947 and by M. H. Rogers and G. N. Lance [35] in 1962 and more recently by D. Dijkstra and P. J. Zandberger [5] in 1977 and M. Lentini and H. B. Keller in 1980 [21]. These recent calculations are concerned with "tracing'out" the branches of the solution set. In particular, these calculations strongly imply the non-unicity of the solution. Formal works on the von Kármán problem have been carried out by M. G. Rogers and G. N. Lance [35], W. G. Cochran [4], H. Ockenden [28] and H. K. Kuiken [19]. Rigorous results on the existence and uniqueness question are not complete. J. B. McLeod [22] has shown existence of a solution for all non-negative values of Ω_∞. In addition, he has shown non-existence for $\Omega_\infty = -1$.

For the two disk problem numerical calculations have been carried out by C. E. Pearson [30]; Lance and Rogers [20]; D. Greenspan [7], D. Schultz and D. Greenspan [36]; G. L.

Mellor, P. J. Chapple and V. K. Stokes [26]; N. D. Nguyen, J. P. Ribault and P. Florent [27]; S. M. Roberts and J. S. Shipman [34]; H. B. Keller and R. K-H. Szeto [13]; L. O. Wilson and N. L. Schryer [42]; G. H. Hoffman [10]; H. J. Pesch and P. Rentrop [31]; M. Kubicek, M. Holodniok, V. Hlaváček [11]; [17], [18]. Formal matched asymptotic expansion methods have been used by A. Watts [40] (who also did numerical calculations), K. K. Tam [39], H. Rasmussen [33], B. J. Matkowsky and W. L. Siegmann [25]. Undoubtably many others have also worked on this problem.

As in the case of the single-disk problem, the rigorous mathematical results for the two disk problem are incomplete. The basic questions of "existence" and "uniqueness" have remained unanswered. S. P. Hastings [9] and A. R. Elcrat [6] have proven existence and uniqueness for large ε. Their arguments are essentially a perturbation about $\varepsilon = \infty$. J. B. McLeod and S. V. Parter [23] considered the special case where $\Omega_0 = -\Omega_1 \neq 0$. They have shown the existence of a solution for all $\varepsilon > 0$ and; for these solutions, they gave a complete discussion of the asymptotic behavior. More recently H. O. Kreiss and S. V. Parter [16] have proven the existence of many "large amplitude" solutions.

Through these 60 years since the basic von Kármán paper and the 30 years since the Batchelor paper the interaction between physically based conjecture, numerical calculations, formal asymptotic expansions and rigorous mathematical results has been intensive. In the remainder of this paper we will discuss several specific questions and describe this interaction. Of course, the view we present is one which is influenced by our own work and interests.

In Section 2 we discuss the counter-rotating case: $\Omega_0 = -\Omega_1 \neq 0$. In Section 3 we discuss the monotone co-rotating case: $0 < \Omega_0 < \Omega_1$, $G'(x,\varepsilon) > 0$. Section 4 describes the results for the case where the "basic" scaling applies

(1.9) $$H(x,\varepsilon) = O(\sqrt{\varepsilon}), \qquad G(x,\varepsilon) = O(1) .$$

Section 5 discusses the case of "order 1" solutions. In Section 6 we turn to the question of "cells". Section 7 describes the existence theory for "large amplitude" solutions. Finally in Section 8 we discuss some more unanswered questions.

2. Counter-Rotating Disks

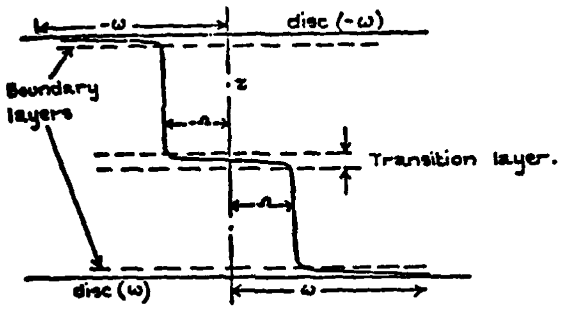
In his 1951 paper [2] G. K. Batchelor gave special attention to the case

$$G(0,\varepsilon) = -1, \qquad G(1,\varepsilon) = 1 \ . \tag{2.1}$$

He suggests that one of the possible solutions the main body of the fluid would be in two parts with different angular velocities - see Figure 1 which is reproduced from [2]. In 1952 K. Stewartson discussed this problem and, using a power series in the Reynolds number

$$R = 1/\varepsilon$$

and obtained a solution in which the core has (essentially) zero angular velocity.



Distribution of angular velocity between the discs. $(\Omega_2 = -\omega,\ d^2\omega/\nu \to \infty)$.

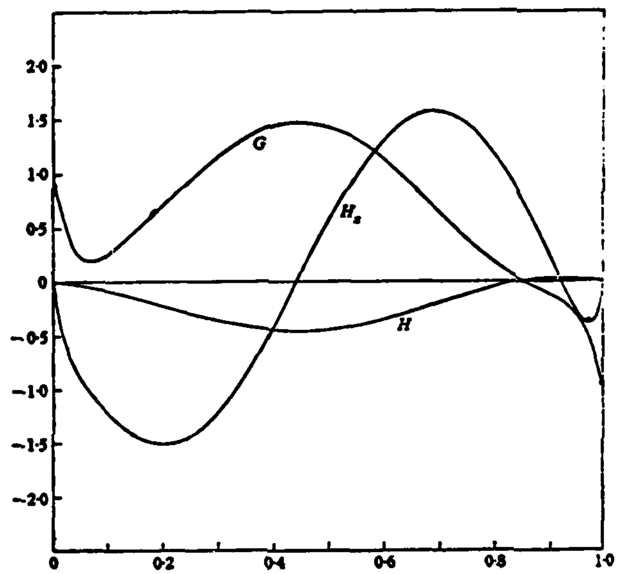
Figure 1

In 1965 C. Pearson [30] computed (numerically) solutions of the steady state problem as the $(t \to \infty)$ limit of a transient problem. His results were startling in that his solutions were not "odd" about $x = 1/2$. That is

$$H(x,\varepsilon) \neq -H(1-x,\varepsilon), \qquad G(x,\varepsilon) \neq -G(1-x,\varepsilon) \ .$$

Thus, as Serrin [37] observed, Pearson's results implied non-uniqueness. Moreover, the Pearson solution had none of the characteristics suggested by either Batchelor or Stewartson (see Figure 2 - taken from Pearson [30]).

In 1974 G. H. Hoffman [10] studied this problem (among others) using a method of computer extension of the Stewartson perturbation series.



Profiles of G, H, H_2, for steady-state motion between two counter-rotating disks with $R = 1000$.

Figure 2

Tam [39] (1969), Rasmussen [33] (1970), and Watts [40] (1974) applied matched asymptotic expansions to this problem.

In 1974 J. B. McLeod and S. V. Parter [23] proved the existence of an "odd" solution (odd about $x = 1/2$). Moreover, they gave a complete asymptotic analysis $(\varepsilon \to 0)$. In particular, $G(x,\varepsilon)$ is monotone, i.e.

(2.2) $$G'(x,\varepsilon) > 0, \quad 0 < x < 1/2.$$

On the interval $[1/2, 1]$ $H(x,\varepsilon)$ is characterized by three points x_1, x_2, x_3 and its negativity. We have (see Figure 3)

(2.3) $$H(x,\varepsilon) < 0, \quad 1/2 < x < 1,$$

(2.4a) $$H'(x,\varepsilon) < 0, \quad 1/2 < x < x_1,$$

(2.4b) $$H'(x,\varepsilon) > 0, \quad x_1 < x < 1,$$

$$0 < H''(x,\varepsilon), \quad \tfrac{1}{2} < x < x_2 , \tag{2.5a}$$

$$H''(x,\varepsilon) < 0, \quad x_2 < x \leq 1 , \tag{2.5b}$$

$$0 < H'''(x,\varepsilon), \quad \tfrac{1}{2} \leq x < x_3 , \tag{2.6a}$$

$$H'''(x,\varepsilon) < 0, \quad x_3 < x \leq 1 . \tag{2.6b}$$

Furthermore, in the core $G(x,\varepsilon)$ is exponentially small while in the boundary layers (at $x = 0$ and $x = 1$) the solution $(H(x,\varepsilon),G(x,\varepsilon))$ is asymptotically the solution of a von Kármán problem with $\Omega_\infty = 0$. Finally, consistent with the remarks above, the solution satisfies the basic scaling (1.9).

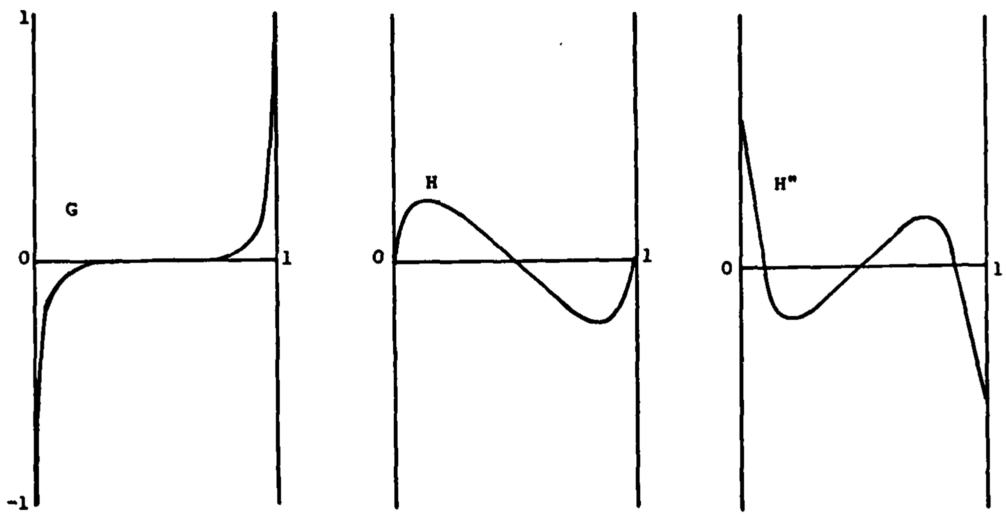


Figure 3

Thus, [23] yielded the existence, for all $\varepsilon > 0$, of an odd Stewartson type solution. Since there was no uniqueness theorem, these results did not exclude the possibility of a Batchelor type solution or a Pearson type solution.

One result of [23] asserts that: If $(H(x,\varepsilon),G(x,\varepsilon))$ is a solution which is odd about $x = \tfrac{1}{2}$ then the condition (2.2) implies the conditions (2.3)-(2.7b) and vice-versa. This result showed that the computational results of [7] were not a good approximation to a solution and led to the improved numerical method described in [36]. In addition, the singular Batchelor solution is an odd solution. His picture seems to indicate that $G' > 0$. If that was his intention, then this result eliminates the possibility of such a singular solution. Indeed, since Batchelor expressed his equations in such a way as to imply the basic scaling (1.9), the singular solution is ruled out by the results of Section 4.

Since 1974 there has been further computational work e.g. [31] and matched asymptotic expansion work, e.g. [25], on this problem.

Finally, the recent results of Kreiss and Parter [16] show that there are many "large amplitude" solutions for ϵ sufficiently small. It seems reasonable that the Pearson solution is a "single hump" large amplitude solution. Watts [40] came to this conclusion on the basis of his work in matched asymptotic expansions. We will have more to say about this in Section 6.

3. Co-Rotating Disks

Let us first consider the case

$$\Omega_0 = \Omega_1 = 1; \quad \text{i.e.,} \quad G(0,\varepsilon) = G(1,\varepsilon) = 1 . \tag{3.1}$$

In this case one sees at least one solution at once

$$H(x,\varepsilon) \equiv 0, \qquad G(x,\varepsilon) \equiv 1 . \tag{3.2}$$

It is not difficult to see that this solution is "stable", i.e. there is no local bifurcation - see [3].

Let $G(1,\varepsilon)$ be fixed, i.e.

$$G(1,\varepsilon) = 1 , \tag{3.3}$$

and let

$$G(0,\varepsilon) = s . \tag{3.4}$$

From the remark above, there is an interval $s_1(\varepsilon) < s < s_2(\varepsilon)$ with $s_1(\varepsilon) < 1 < s_2(\varepsilon)$, so that there is a solution of (1.1), (1.2), (1.8a), (1.8b), (3.3) and (3.4) for $s \in (s_1(\varepsilon), s_2(\varepsilon))$. This remark is the basis for a numerical method studied by M. Kubicek, M. Holodniak and V. Hlaváček [18] and the work of J. Cerutti [3]. However, we have no knowledge of the behavior of $s_1(\varepsilon)$, $s_2(\varepsilon)$ as $\varepsilon \to 0+$.

The formal asymptotic work of Stewartson [38] and Watts [40] indicates that there may be other solutions of this problem which also satisfy the basic scaling (1.9). Watts suggests there is a solution very much like the solution obtained for the counter-rotating case. In this case we expect

$$H(x,\varepsilon) = -H(1-x,\varepsilon) \tag{3.5a}$$

$$G(x,\varepsilon) = G(1-x,\varepsilon) \tag{3.5b}$$

with $G(x,\varepsilon)$ exponentially small in the core and $H(x,\varepsilon)$ having a "shape" similar to the shape described by (2.3)-(2.6b). Some of our preliminary work indicates that such solutions do indeed exist for ε small enough. However, we conjecture that if such solutions exist, then

$$G(0,\varepsilon) < 0 . \tag{3.6}$$

In the general co-rotating case

(3.7) $$0 \leq G(0,\varepsilon) < G(1,\varepsilon) .$$

Batchelor suggested that the angular velocity would be monotone, i.e.

(3.8) $$G'(x,\varepsilon) > 0, \quad 0 \leq x \leq 1 .$$

In fact, this is false. McLeod and Parter [24] have shown:

Let $G(0,\varepsilon)$, $G(1,\varepsilon)$ be fixed and satisfy (3.7). Then there is an $\tilde{\varepsilon} = \tilde{\varepsilon}(\Omega_0,\Omega_1)$ such that if $0 < \varepsilon \leq \tilde{\varepsilon}$ and if there is a solution of (1.1), (1.2), (1.8a), (1.8b) and (1.8c), the inequality (3.8) is false.

Once more, the results of Kreiss and Parter [16] show that for ε small enough there are (many) solutions to this problem.

4. The Basic Scaling

Many of the authors dealing with this problem have assumed the basic scaling (1.9);

$$(4.1)\qquad |H(x,\varepsilon)| \leq \sqrt{\varepsilon}\, B, \qquad |G(x,\varepsilon)| \leq B$$

and employed the change of variables (1.4a), (1.4b). Thus, one has the equations (1.5), (1.6) on the larger interval $[0,1/\sqrt{\varepsilon}]$. From the point-of-view of computation this leads to regular problems - albeit on a large interval - which is desirable particularly when employing the "shooting method" see [20], [26]. From the intuitive point-of-view as well as from the matched asymptotic viewpoint it is reasonable to assume that the solution in the boundary layers - at both disks - behaves like a solution of the von Kármán problem which then "matches" with a core solution. In fact, this approach was used by Watts [40], Tam [39] and Rasmussen [33] - and implicitly by Stewartson.

Within this context, both Batchelor and Stewartson assumed that: in the core, $\delta < x < 1 - \delta$,

$$(4.2)\qquad g(\xi,\varepsilon) = G(x,\varepsilon) \to G_\infty, \quad \text{a constant} .$$

They considered two types of solutions

Batchelor: In addition to (4.2) we have

$$(4.3)\qquad G_\infty \neq 0, \quad h'(\xi,\varepsilon) \to 0 ,$$

i.e., the core rotates as a rigid body.

Stewartson:

$$(4.4)\qquad G_\infty = 0 .$$

Both agreed that the Batchelor type solution would appear when

$$0 \leq G(0,\varepsilon), \qquad 0 \leq G(1,\varepsilon) ,$$

i.e., the co-rotating case. Stewartson suggested that (4.4) would occur when

$$G(0,\varepsilon)G(1,\varepsilon) < 0 .$$

Solutions of this type have been obtained both numerically and via matched asymptotic expansions.

In fact, if the basic scaling holds, then - in some sense (4.2) must hold. This fact is contained in the following results of Kreiss and Parter [14].

Theorem 4.1 (see Lemma 3.3, Theorem 3.1 and Theorem 4.1 of [14]). Let δ, $0 < \delta < 1/4$ be given. Let $(H(x,\varepsilon_n),G(x,\varepsilon_n))$ be a sequence of solutions of (1.1), (1.2), (1.8a), (1.8b), (1.8c) which satisfy (4.1) for some constant B. Then there is an $\varepsilon(\delta)$ and an $M(\delta)$ such that; if $0 < \varepsilon_n \leq \varepsilon(\delta)$ then for $0 < \delta \leq x \leq 1 - \delta < 1$ we have

$$(4.5) \qquad \left| \tfrac{1}{2} G^2(x,\varepsilon_n) - \mu(\varepsilon_n) \right| \leq M(\delta)(1 + B)\varepsilon_n^{\frac{1}{128}} \to 0 \quad \text{as} \quad \varepsilon_n \to 0+ .$$

Obviously one can extract a subsequence $\varepsilon_n' \to 0+$ so that

$$(4.6a) \qquad \mu(\varepsilon_n') \to \bar{\mu} \geq 0 ,$$

$$(4.6b) \qquad G(x,\varepsilon_n') \to \pm 2\sqrt{\bar{\mu}} = G_\infty .$$

Suppose this has been done. If

$$(4.7) \qquad \bar{\mu} > 0 ,$$

then there is a constant a such that

$$(4.8a) \qquad H(x,\varepsilon_n')/\sqrt{\varepsilon_n'} \to a, \qquad \delta < x < 1 - \delta .$$

In fact, both

$$(4.8b) \qquad \left| \tfrac{1}{2} G^2(x,\varepsilon_n') - \bar{\mu} \right|, \qquad \left| \frac{1}{\sqrt{\varepsilon_n'}} H(x,\varepsilon_n') - a \right|$$

are exponentially small (in ε_n').

Remark: Two important points must be made. First; this is an asymptotic theorem, there is no assertion of existence of solutions. There is only the statement that if such solutions exist, this result describes their asymptotic behavior. Secondly, the statement that in the boundary-layer the solution is essentially the solution of a von Kármán is suggested but this discussion is not entirely complete.

The case when $G_\infty = 0$ is more complicated. A partial discussion is given in Section 6.

5. Order One Solutions

While the basic scaling (1.9) has many attractions there is another plausible scaling. We assume

$$|H(x,\varepsilon)| + |H'(x,\varepsilon)| + |G(x,\varepsilon)| \leq B . \tag{5.1}$$

If this bound holds then the physical velocities (q_r, q_θ, q_x) are bounded in any cylinder $r \leq R$. However, in order to guarantee that we are not dealing with the case described earlier, we insist that $H(x,\varepsilon)$ be truly of order 1. Specifically, we assume there is a point x_0, $0 < x_0 < 1$ and a constant $\delta > 0$ so that

$$|H(x_0,\varepsilon)| \geq \delta > 0 . \tag{5.2}$$

In his work on this problem with matched asymptotic expansions [33] Rasmussen had trouble in the case where H and G are of the same order. It has been suggested that this problem is involved with the intrinsic difficulties of Ackerberg-O'Malley Resonance [1].

In fact, the matter is quite simple: (essentially) there are no such solutions!!

The argument is in two parts. If (5.1) holds then there is a subsequence $\varepsilon_n' \to 0+$ and a function $\bar{H}(x)$ so that

$$H(x,\varepsilon_n') \to \bar{H}(x) \text{ uniformly on } [0,1] . \tag{5.3}$$

Further, it can be arranged that (5.2) takes the form

$$H(x_0,\varepsilon_n') \geq \delta > 0 . \tag{5.2'}$$

Let $\beta > x_0$ be the first point greater than x_0 at which $\bar{H}(\beta) = 0$. Then

$$\bar{H}'(\beta) = 0 . \tag{5.3}$$

This result is explicitly given as Theorem 4.2 of [15].

It now follows that $\bar{H}(x) \geq 0$ and is of the following form: There are N numbers σ_j, $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N < \sigma_{N+1} = 1$ and, on the interval $[\sigma_j, \sigma_{j+1}]$, $j = 0,1,\ldots,N$ either $\bar{H}(x)$ is a quadratic or $\bar{H}(x)$ is of the form

$$\bar{H}(x) = A_j[1 - \cos\tau_j(x - \sigma_j)] , \tag{5.4a}$$

where

$$\tau_j = \frac{2\pi}{\sigma_{j+1} - \sigma_j} . \tag{5.4b}$$

Finally, while this result is never explicitly stated in [15], the argument given in [16] implies that

$$|G(0,\varepsilon_n')| + |G(1,\varepsilon_n')| = O((\varepsilon_n')^{2/3}) \quad (5.5)$$

Therefore, (5.5) is a necessary condition for the existence of "order 1" solutions.

6. Cells

In [26] Mellor, Chapple and Stokes introduced the concept of a cell and computed several multi-cell solutions. A cell is the region between successive zeros x_1, x_2 of $H(x,\varepsilon)$. This is a region [in (r,θ,x) space] or cell in which a portion of the fluid is "trapped", i.e. the fluid cannot cross the boundaries $x = x_1$, $x = x_2$. Unfortunately this definition is not "tight" enough. It allows for the existence of cells in the boundary layers which are lost as $\varepsilon \to 0+$.

For this reason we have adapted the following approach: Let there be a number ρ so that; if

(6.1a) $$h(x,\varepsilon) = \varepsilon^{\rho} H(x,\varepsilon),$$

and in the interior of $(0,1)$

(6.1b) $$h(x,\varepsilon_n) \to \bar{h}(x), \quad \text{as} \quad \varepsilon_n \to 0 \; ,$$

Definition: A "cell" is an interval (α,β) with $0 \le \alpha < \beta \le 1$ such that

(6.2a) $$\text{either} \quad \alpha = 0 \quad \text{on} \quad \bar{h}(\alpha) = 0, \quad \text{and}$$

(6.2b) $$\text{either} \quad \beta = 1 \quad \text{or} \quad \bar{h}(\beta) = 0, \quad \text{and}$$

(6.2c) $$|\bar{h}(x)| > 0, \qquad \alpha < x < \beta \; .$$

The solution obtained in [23] in the counter rotating case has $\rho = -\frac{1}{2}$ and leads to two oscillating cells. The basic result is

Theorem: (see Section 5 of [15]). Suppose there are at least two cells, (α_1,β_1), (α_2,β_2) with

(6.3a) $$\beta_1 \le \alpha_2$$

and, these cells "oscillate", that is

(6.3b) $$\bar{h}(x) > 0, \qquad \alpha_1 < x < \beta_1 \; ,$$

(6.3c) $$\bar{h}(x) < 0, \qquad \alpha_2 < x < \beta_2 \; .$$

Then $\bar{h}(x)$ has most 4 cells. Moreover, $\bar{h}(x)$ is a piecewise quadratic function with at most two distinct pieces. That is, there is a point $x_0 \in [0,1]$ - Note: x_0 can be 0 or 1 - and $\bar{h}(x)$ has the following form

(6.4a) $\quad \bar{h}(x) \in C', \qquad 0 \leq x \leq 1,$

(6.4b) $\quad \bar{h}(x) = a_1x^2 + a_2x + a_3, \qquad 0 \leq x \leq x_0,$

(6.4c) $\quad \bar{h}(x) = b_1x^2 + b_2x + b_3, \qquad x_0 \leq x \leq 1.$

The function $g(x)$ is exponentially small in any strict interior subinterval of the two intervals $(0,x_0)$, $(x_0,1)$.

In the case of the basic scaling, then $\rho = -1/2$. Thus we see that: in the case of the basic scaling, if $\varepsilon^{-1/2}H(x,\varepsilon)$ is convergent, then it is a piecewise quadratic with at most two pieces.

Of course, Mellor, Chapple and Stokes used the basic scaling and $\rho = -1/2$ in their case.

In order to complete the discussion of the basic scaling we would need to know that

$$\varepsilon^{-1/2}H(x,\varepsilon)$$

must be convergent when $G_\infty = 0$.

7. Large Amplitude Solutions

In [40] Watts introduced the notion of a large amplitude solution $(H(x,\varepsilon),G(x,\varepsilon))$. He obtained a formal asymptotic expansion for a solution with one "hump", i.e., the function $H(x,\varepsilon)$ had the form

$$H(x,\varepsilon) = A(\varepsilon)[1 - \cos 2\pi x], \qquad \delta < x < 1 - \delta ,$$

where

$$A(\varepsilon) \sim \varepsilon^{-2} .$$

The function $G(x,\varepsilon)$ had the form

$$G(x,\varepsilon) = \pm(2\pi)H(x,\varepsilon) .$$

Watts also states that he sees no reason why an n-hump solution cannot be constructed by the same method. It seems he also obtained the relation (7.6a) (see Figure 4 - taken from Watts [40]). He makes no mention of the quantity $\breve{\tau}$.

The main result of the most recent paper of Kreiss and Parter [16] is Theorem 7.1 (see Theorem II of [16]). Let $n > 1$ be a given integer. Let s be a fixed real number. Then, there is an $\bar{\varepsilon} = \bar{\varepsilon}(n,s)$ such that; for $0 < \varepsilon < \bar{\varepsilon}$ there exists a solution $(H(x,\varepsilon),G(x,\varepsilon))$ of (1.1), (1.2), (1.8a), (1.8b), (1.8c) with

$$\Omega_0 = s, \quad \Omega_0 = 1 . \tag{7.1}$$

This solution may be described as follows. There are exactly $(n + 1)$ numbers

$$0 = \sigma_0(\varepsilon) < \sigma_1(\varepsilon) < \cdots < \sigma_{n-1}(\varepsilon) < \sigma_n(\varepsilon) = 1 \tag{7.2a}$$

at which $H(x,\varepsilon)$ has its relative minima, i.e.,

$$H'(\sigma_j(\varepsilon),\varepsilon) = 0, \qquad H''(\sigma_j(\varepsilon),\varepsilon) > 0 . \tag{7.2b}$$

Between the $\sigma_j(\varepsilon)$ the function $H(x,\varepsilon)$ is essentially positive. That is, for $\delta > 0$, $2\delta < \sigma_{j+1} - \sigma_j$ we have - for small ε -

$$H(x,\varepsilon) > 0, \ \sigma_j(\varepsilon) + \delta < x < \sigma_{j+1}(\varepsilon) - \delta . \tag{7.3}$$

The numbers $\sigma_j(\varepsilon)$ satisfy

$$\lim_{\varepsilon=0} \frac{2\pi}{\sigma_{j+1}(\varepsilon) - \sigma_j(\varepsilon)} = \xi_j = \xi_0|\breve{\tau}|^j \tag{7.4}$$

where $\breve{\tau}$ is a fixed number which will be described in the Appendix. The number ξ_0 is determined from this relationship and (7.2a). We have

$$\frac{1}{2\pi} = \sum_{j=0}^{n-1} \frac{\sigma_{j+1}(\varepsilon) - \sigma_j(\varepsilon)}{2\pi} + \frac{1}{\xi_0}\left(\frac{\theta^n - 1}{\theta - 1}\right)$$

where

$$\theta = |\breve{\tau}|^{-1} .$$

Thus

(7.5)
$$\xi_0 = 2\pi\left(\frac{1 - |\breve{\tau}|^n}{1 - |\breve{\tau}|}\right) \frac{1}{|\breve{\tau}|^{n-1}} .$$

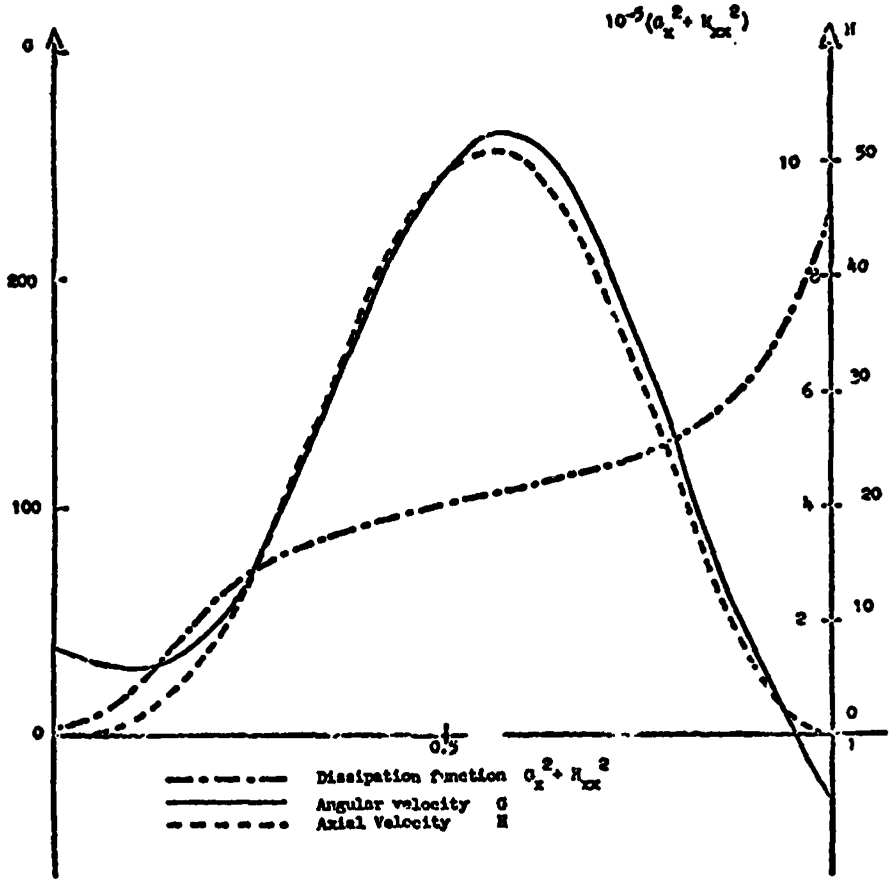


Figure 4. Large Amplitude Solution Re = 57 s = 0.75

The function $G(x,\varepsilon)$ has at least n nodal zeros; $0 < \gamma_1(\varepsilon) < \gamma_2(\varepsilon) < \cdots < \gamma_n(\varepsilon) < 1$. Moreover

(7.6a) $$\gamma_j(\varepsilon) < \sigma_j(\varepsilon), \quad \sigma_j(\varepsilon) - \gamma_j(\varepsilon) = O(\varepsilon) .$$

If $s \neq 0$ and

$$\text{sign } s = (-1)^{n+1} .$$

Then $G(x,\varepsilon)$ has $(n + 1)$ zeros. The additional zero, $\gamma_0(\varepsilon)$ satisfies

(7.6b) $$0 < \gamma_0(\varepsilon) = O(\varepsilon) .$$

Finally

(7.7a) $$\|H\| \approx \varepsilon^{-2}, \quad \|G\| \approx \varepsilon^{-2} ,$$

and, on the interval $\sigma_j + \delta < x < \sigma_{j+1} - \delta$ we have

(7.7b) $$\varepsilon^2 G(x,\varepsilon) \approx \varepsilon^2(-1)^{j+1}\xi_j H(x,\varepsilon) ,$$

(7.7c) $$\varepsilon^2 H(x,\varepsilon) \approx A_j[1 - \cos\xi_j(x - \sigma_j)] .$$

Before going on to a sketch of the proof, it is worthwhile to see what are some of the consequences of this theorem. When discussing "branches" of solution of homogeneous second order equations it is useful to characterize solutions by the number of interior zeros. Since $G(x,\varepsilon)$ satisfies such a homogeneous second order equation, let us characterize the solution pair $\langle H(x,\varepsilon),G(x,\varepsilon)\rangle$ by the zeros of $G(x,\varepsilon)$.

Case 1: $s > 0$. For every even $\bar{n} \geq 2$ there are at least two solutions $\langle H,G\rangle, \langle \breve{H},\breve{G}\rangle$ with $G(x,\varepsilon),\breve{G}(x,\varepsilon)$ having exactly $\bar{n}$ interior zeros and

$$H(x,\varepsilon) > 0, \quad \breve{H}(x,\varepsilon) > 0, \quad \text{(essentially)} .$$

Proof: Let $n = \bar{n}$. Since

$$\text{sign } s = (-1)^{\bar{n}} > 0$$

$$\text{sign } s \neq (-1)^{n+1}$$

the function $G(x,\varepsilon)$ of the solution pair $\langle H,G\rangle$ described in Theorem 7.1 has exactly n interior zeros. However, the function $\breve{G}(x,\varepsilon)$ of the solution pair $\langle \breve{H},\breve{G}\rangle$ associated with $n = \bar{n} - 1$ also has exactly $\bar{n} = (n - 1) + 1$ interior zeros.

Case 2: $s < 0$. For every odd $\bar{n} \geq 3$ there are at least two solutions $\langle H,G\rangle, \langle \breve{H},\breve{G}\rangle$ with $G(x,\varepsilon)$, $\breve{G}(x,\varepsilon)$ having exactly $\bar{n}$ interior zeros and

$$H(x,\varepsilon) > 0, \quad \breve{H}(x,\varepsilon) > 0, \quad \text{(essentially)} .$$

If $\bar{n} = 1$ there is at least one solution $\langle H,G \rangle$ with $G(x,\varepsilon)$ having exactly one interior zero while $H(x,\varepsilon)$ is essentially positive.

<u>Proof</u>: Let $n = \bar{n}$. Then

$$-1 = \text{sign } s \neq (-1)^{n+1} = 1 .$$

Then, for the solution $\langle H,G \rangle$ described in Theorem 7.1 $G(x,\varepsilon)$ has exactly $n = \bar{n}$ interior zeros. If $\bar{n} \geq 3$ then let $n = \bar{n} - 1$. Since

$$-1 = \text{sign } s = (-1)^{n+1} = (-1)^{\bar{n}}$$

the function $\breve{G}(x,\varepsilon)$ of the solution $\breve{H},\breve{G}$ associated with $n = \bar{n} - 1$ has exactly $n + 1 = \bar{n}$ interior zeros.

<u>Case 3</u>: $s = 0$. For every $\bar{n} \geq 1$ (even or odd) there is at least one solution $\langle H(x,\varepsilon),G(x,\varepsilon) \rangle$ with $G(x,\varepsilon)$ having exactly $\bar{n}$ interior zeros while $H(x,\varepsilon)$ is essentially positive.

<u>Proof</u>: Let $n = \bar{n}$ and let $\langle H,G \rangle$ be the solution described in Theorem 7.1.

This theorem is proven by first obtaining an $O(1)$ solution with boundary values $G(0,\bar{\varepsilon})$, $G(1,\bar{\varepsilon})$ which are $O(\varepsilon^{2/3})$. These "pathological" solutions are obtained from a "shooting" argument.

The arguments and results of [15] are used to show that when $H(x,\varepsilon) \geq k\varepsilon^{2/3}$ then $\langle H(x,\varepsilon),G(x,\varepsilon) \rangle$ must have the form

$$H(x,\varepsilon) = \frac{h_2}{\tau^2}(1 - \cos\tau(x - \sigma)) , \tag{7.8a}$$

$$G(x,\varepsilon) = \tau H(x,\varepsilon) . \tag{7.8b}$$

On the other hand, when

$$H(x,\varepsilon) = O(\varepsilon^{2/3}) \tag{7.9}$$

one employs the change of variables

$$\xi = \frac{x - x_0}{\varepsilon^{1/3}} , \tag{7.10a}$$

$$h(\xi,\varepsilon) = \varepsilon^{-2/3}H(x,\varepsilon), \quad g(\xi,\varepsilon) = \varepsilon^{-2/3}G(x,\varepsilon) . \tag{7.10b}$$

The functions $\langle h,g \rangle$ satisfy the equations

$$h''' + hh'' + \frac{1}{2}\varepsilon^{2/3}g^2 - \frac{1}{2}(h')^2 = \mu/\varepsilon^{2/3} , \tag{7.11a}$$

$$g'' + hg' - h'g = 0 . \tag{7.11b}$$

The initial values are chosen so that

$$|\mu| < \kappa\varepsilon, \quad |\mu/\varepsilon^{2/3}| < \kappa\varepsilon^{1/3} . \tag{7.12}$$

Now it is not difficult to see that $h(\xi,\varepsilon)$ converges to a quadratic function $\bar{h}(\xi)$ of the form

$$\bar{h}(\xi) = \frac{1}{2} h_2(\xi - \xi_1)^2, \quad h_2 \text{ a constant} . \tag{7.13a}$$

Thus, $g(\xi,\varepsilon)$ converges to a function $\bar{g}(\xi)$ which satisfies

$$\bar{g}'' + \bar{h}\bar{g}' - \bar{h}'\bar{g} = 0 . \tag{7.13b}$$

The final result depends on an elementary degree theory argument and an analysis of the solutions of

$$\bar{g}'' + \xi^2\bar{g}' - 2\xi\bar{g} = 0 . \tag{7.14}$$

The facts about this equation are described in the Appendix.

8. Comments and Questions

Despite all we now know about the solutions of (1.1), (1.2), (1.8a), (1.8b), (1.8c), there are many interesting questions still unanswered.

Question 1: It is not difficult to see that the solutions obtained by Hastings [9] and Elcrat [6] for large $\varepsilon >> 1$ can be obtained by an iterative procedure with ε as a parameter. Thus, for ε large we have a curve of solutions.

The first question is: keeping Ω_0, Ω_1 fixed and varying ε downward, how far can these continua of solutions be continued??

It is reasonable to assume that these solutions exist for all $\varepsilon > 0$. The next question is; if that is so, do these solutions $\langle H,G \rangle$ satisfy the basic scaling (1.9), i.e.

(8.1) $$H = O(\sqrt{\varepsilon}), \qquad G = O(1) \quad ??$$

Question 2: With Ω_0, Ω_1 fixed, are there families of solutions which satisfy (8.1)?? Recall that when $\Omega_0 = -\Omega_1$ the solutions obtained by McLeod-Parter [23] do indeed satisfy (8.1). And of course, the trivial solution (3.2) satisfies (8.1). The computational evidence suggests that there are such solutions.

Question 3: If there are solutions satisfying (8.1), are they unique?? The computational evidence suggests that there are many solutions. If there are many solutions - how does one characterize the possible values G_∞ of (4.2). The results of Rasmussen [33] and the conjecture of Stewartson [38] suggests that whenever

$$\Omega_0 \, \Omega_1 < 0$$

we must have

$$G_\infty = 0 \, .$$

Question 4: Do the large amplitude solutions lie on continua of solutions? In particular, given s and $\bar{n} > 2$, is there a continuum of solutions which exist for $\varepsilon < \bar{\varepsilon}(\bar{n},s)$ which contains the pair $\langle H,G \rangle$, $\langle \check{H},\tilde{G} \rangle$ of solutions with $G(x,\varepsilon)$ and $\tilde{G}(x,\varepsilon)$ having exactly $\bar{n}$ interior zeros described in Section 7??

Question 5: Are there "other" solutions?? That is; are there families of solutions which do not satisfy (8.1) other than those large amplitude solutions $(H \sim \varepsilon^{-2}, G \sim \varepsilon^{-2})$ found in [16]?? Of course, for each solution $-H(1-x,\varepsilon), G(1-x,\varepsilon)$ is again a solution. Hence there are related large amplitude solutions with $H(x,\varepsilon) < 0$ (essentially).

Question 6: The function (first discussed by McLeod [22])

$$\phi(x,\varepsilon) = [(G')^2 + (H'')^2]$$

plays an important role in much of the analysis - see [22], [23], [14]. It is characterized by the fact that there is a unique point $\gamma = \gamma(\varepsilon) \in [0,1]$ at which $\phi(x,\varepsilon)$ has a minimum. Furthermore, if $\phi(x,\varepsilon)$ has a relative minimum, it is the minimum. Watts [40] observes that - asymptotically - for the "solutions" which he obtain which also satisfy (8.1),

$$0 < \text{Lim}\ \gamma(\varepsilon) < 1 \tag{8.2}$$

while in the case of his large amplitude solutions

$$\text{Lim}\ \gamma(\varepsilon) = 0 \text{ or } 1 \ . \tag{8.3}$$

It is not difficult to show that the large amplitude solutions constructed in [16] do indeed satisfy (8.3). The question is: does $\text{Lim}\ \gamma(\varepsilon)$ characterize the "size" of all solutions?

Question 7: The negative results of McLeod and Parter in Section 3 and the monotonicity of $G(x,\varepsilon)$ for the counter-rotating case leads to the following observation and question. For $\Omega_0 = -\Omega_1 = -1$ and all $\varepsilon > 0$ there is a solution $\langle H,G \rangle$ with

$$G'(x,\varepsilon) > 0 \ .$$

For, $0 \leq \Omega_0 < \Omega_1$ that statement cannot be true. Therefore, the question is: if there is a number $\bar{a}$, $-1 < \bar{a} < 0$ determined such that; if $\Omega_1 = 1$, and

$$-1 \leq \Omega_0 < \bar{a} \tag{8.2}$$

then, for all $\varepsilon > 0$ there is a solution $\langle H,G \rangle$ with $G'(x,\varepsilon) > 0$. If there is no such number, then $\Omega_0 = -1$ is a very special case indeed.

The final question is a very large one. Given a solution how can one determine its time-dependent stability?

Appendix

A key part of the argument leading to results of Section 7 - i.e. the results of [16] - is the analysis of the solutions of

(A.1) $$g'' + \delta x^2 g' - 2\delta x g = 0 .$$

We sketch this analysis.

We can restrict ourselves to the case $\delta = 1$. Let $g(x;1)$ be a solution of (A.1) with $\delta = 1$. Then, for any $\delta > 0$, a direct calculation shows that

(A.2) $$Y(x;\delta) = g(\delta^{1/3}x;1)$$

is a solution of (A.1) with this value of δ.

For the remainder of this discussion we have $\delta = 1$.

Using the W.K.B.J. method (see Chapter 6 of [29] and the method described by Wasow in [41, pp. 52-61] we see that there are two linearly independent solutions $g_1(x)$, $g_2(x)$ and

(A.3a) $$g_1(x) \sim x^2(1 + 2/3x^2), \quad x \to -\infty ,$$

(A.3b) $$g_1'(x) \sim 2x(1 + 2/3x^2), \quad x \to -\infty ,$$

(A.3c) $$g_1'' \sim 2, \quad x \to -\infty ,$$

(A.3d) $$g_2(x) \sim x^{-4}\exp\left[-\frac{x^3}{3}\right], \quad x \to -\infty .$$

Similarly, there are two linearly independent solutions $\varphi_1(x)$, $\varphi_2(x)$ and

(A.4a) $$\varphi_1(x) \sim x^{-4}\exp\left[-\frac{x^3}{3}\right], \quad x \to +\infty ,$$

(A.4b) $$\varphi_2(x) \sim x^2, \quad x \to +\infty .$$

Since the function $g_1(x)$ can be written as a linear combination of $\varphi_1(x)$ and $\varphi_2(x)$ we see that there is a unique constant $\breve{\tau}$ such that

(A.5) $$g_1(x)/x^2 \to \breve{\tau}, \quad x \to +\infty .$$

Of course, this $\breve{\tau}$ is the quantity of Section 7.

We don't need to know much about the functions $\varphi_1(x)$, $\varphi_2(x)$. It suffices that

(A.6) $$\varphi_1(0) \neq 0 .$$

This elementary result follows almost immediately from the maximum principle or the representations

(A.7a) $$\frac{d}{dx}\{\varphi_1' \exp[\frac{x^3}{3}]\} = 2x\varphi_1 \exp[\frac{x^3}{3}] \;,$$

(A.7b) $$\frac{d}{dx}\{\varphi_1'' \exp[\frac{x^3}{3}]\} = 2\varphi_1 \exp[\frac{x^3}{3}] \;,$$

and the fact

(A.8) $$\varphi_1''(0) = 0 \;.$$

Our major interest centers on the function $g_1(x)$. The basic facts are:

(A.9a) $$g_1'(x) < 0, \qquad -\infty < x < \infty \;.$$

There is a value, say $\bar{g}$, at which $g_1(\bar{g}) = 0$. This unique zero can be estimated by

(A.9b) $$-1 < \bar{g} < 0 \;.$$

Finally

(A.9c) $$\check{\tau} < 0 \;.$$

These results are obtained by a detailed argument based on elementary considerations, the maximum principle, the oscillation theorem and the series expansion of the two functions $Y_1(x)$, $Y_2(x)$ which satisfy (A.1) - with $\delta = 1$ - and also satisfy

(A.10a) $$Y_1(0) = 0, \qquad Y_1'(0) = -1 \;,$$

(A.10b) $$Y_2(0) = -1, \qquad Y_2'(0) = 0 \;.$$

We remark that it is equally easy to obtain these results by rigorous, careful, numerical computation. In fact, computations by Jerry Browning of NCAR indicate

$$\check{\tau} = -2 \;.$$

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Consider solutions $(H(x,\varepsilon), G(x,\varepsilon))$ of the von Kármán equations for the swirling flow between two rotating coaxial disks

1.1) $\varepsilon H^{iv} + HH''' + GG' = 0$,

1.2) $\varepsilon G'' + HG' - H'G = 0$.

In this survey we describe much of the activity of the past 30 years - involving physical conjecture, numerical computation, asymptotic expansions and rigorous mathematical results. In particular we focus on the questions of existence and nonuniqueness, monotonicity, and "scaling".

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